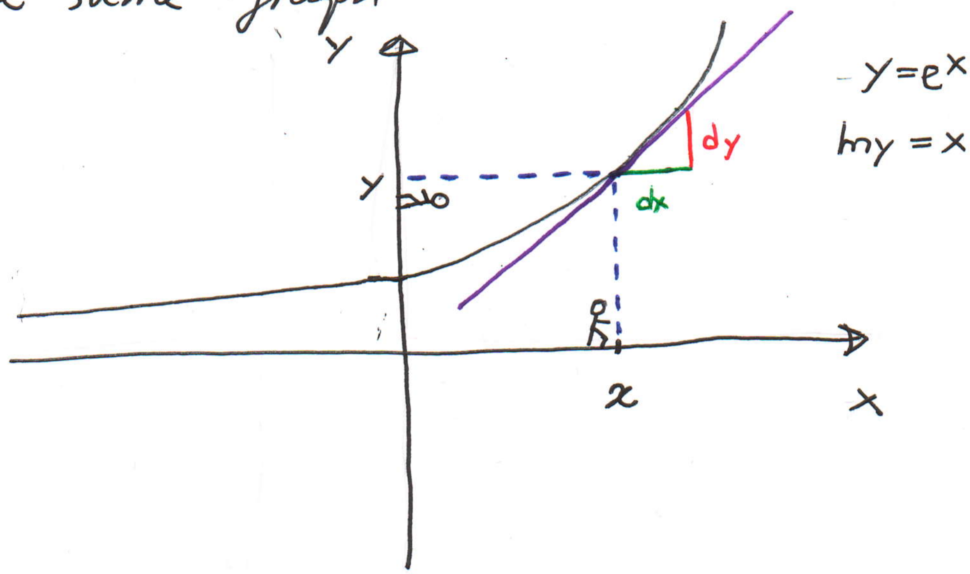
Derivatives of Logarithmic and inverse trigonometric functions Lecture 5

In the previous lecture we have seen that a function and its inverse are two sides of the same coin. Knowing the derivative of a function instantly lets us figure the derivative of the function's inverse.

For example $f(x)=e^x$ and $f^{-1}(x)=\ln x$ are seen on the same graph



Astronaut at point x sees $y=e^x$ and measures slope $\frac{dy}{dx}=e^x$. Astronaut y sees $x=\ln y$ and measures slope $\frac{dx}{dy}=\frac{1}{e^x}$ (why?). Thus in terms of y

$$\frac{d}{dy}(\ln y) = \frac{dx}{dy} = \frac{1}{e^x} = \frac{1}{y}.$$

We may also use implicit differentiation:

Ex. Find $\frac{d}{dx}\ln x$.

Solution: $y = \ln x \Rightarrow e^y = e^{\ln x} = x$

$e^y \cdot y' = 1 \Rightarrow y' = \frac{1}{e^y} = \frac{1}{x}.$

Thus $\frac{d}{dx}\ln x = \frac{1}{x}$.

Ex. Find $\frac{d}{dx}\log_a x$

Solution: $y = \log_a x \Rightarrow a^y = x$. Now

$(a^y)' = \left(e^{\ln(a^y)}\right)' = \left(e^{y\ln a}\right)' = e^{y\ln a}\ln a\, y'$

$= a^y \ln a\, y' = (x)' = 1$.

Thus $a^y \ln a\, y' = 1$, or $y' = \frac{1}{a^y \ln a} = \frac{1}{x\ln a}$.

Logarithms are a useful tool for computing derivatives involving products, quotients, and exponents.

Ex. Compute $\frac{d}{dx}\frac{\left(\sqrt[7]{x^4+25}\right)(x+1)^8}{x^6(x^2+1)^{100}e^{5x}}$

Solution: We may work with a multiple applications of product rules, quotient rules, etc. or we recall the following:

① $\ln(M\cdot N) = \ln(M) + \ln(N)$

② $\ln\left(\frac{M}{N}\right) = \ln(M) - \ln(N)$

③ $\ln(M^r) = r\ln(M)$

Set $y = \dfrac{(x^4+25)^{1/7}(x+1)^8}{x^6(x^2+1)^{100}e^{5x}}$ Then

$\ln y = \ln\left(\dfrac{(x^4+25)^{1/7}(x+1)^8}{x^6(x^2+1)^{100}e^{5x}}\right)$

$= \frac{1}{7}\ln(x^4+25) + 8\ln(x+1) - 6\ln(x) - 100\ln(x^2+1)$

$- 5x\ln e = \frac{1}{7}\ln(x^4+25) + 8\ln(x+1) - 6\ln x -$

$-100\ln(x^2+1) - 5x$

Using implicit differentiation we obtain

$\frac{1}{y}y' = \frac{1}{7}\cdot\frac{4x^3}{x^4+25} + 8\frac{1}{x+1} - 6\frac{1}{x} - 100\cdot\frac{2x}{x^2+1}$

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so $y' = y\left(\frac{1}{7}\cdot\frac{4x^3}{x^4+25} + 8\frac{1}{x+1} - 6\frac{1}{x} - \frac{200x}{x^2+1} - 5\right)$

or $y' = \dfrac{(x^4+25)^{\frac{1}{7}}(x+1)^8}{x^6(x^2+1)^{100}e^{5x}}$

<u>Ex.</u> Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

<u>Solution:</u> $\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$

so $\dfrac{1}{y}\cdot y' = \dfrac{3}{4}\dfrac{1}{x} + \dfrac{1}{2}\dfrac{2x}{x^2+1} - 5\dfrac{3}{3x+2}$

$$y' = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$y' = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).$$

<u>Remark:</u> Technically speaking, when we apply $\ln$ to y, it is implicit that $y>0$.

<u>Ex.</u> We have remarked that $\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$ for every $\alpha \in \mathbb{R}$. The observant student will notice that we have established the validity of power rule only when $\alpha = 0, 1, 2, 3 \ldots$

what do expressions like x^π even mean?

x^α may be defined as $e^{\alpha \ln x}$. Since $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots +$

$e^\pi = 1 + \pi + \frac{\pi^2}{2} + \frac{\pi^3}{6} + \ldots +$ is unambiguous. Only elementary operations are being used.

$$\frac{d}{dx}(x^{\alpha}) = \frac{d}{dx}\left(e^{\alpha \ln x}\right) = e^{\alpha \ln x} \cdot \frac{\alpha}{x} = x^{\alpha} \cdot \frac{\alpha}{x}$$

$$= \alpha x^{\alpha-1} \quad \left(\text{because } e^{\alpha \ln x} \cdot \frac{\alpha}{x} = \alpha e^{\alpha \ln x} \cdot e^{-\ln x} = \right.$$

$$\left. = \alpha e^{(\alpha-1)\ln x} = \alpha x^{\alpha-1}\right).$$

<u>Ex.</u> Find the derivative

(a) $y = \ln|x|$

(b) $y = \ln|\sin x|$

(c) $y = \log_3(e^x)$

(d) $y = \sin x \ln(5x)$

(e) $y = \log_5(xe^x)$

(f) $y = \ln \dfrac{(2x+1)^5}{\sqrt{x^2+1}}$

<u>Solution:</u>

(a) $y = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

$$\left.\frac{dy}{dx}\right|_{x>0} = \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\left.\frac{dy}{dx}\right|_{x<0} = \frac{d}{dx}\left(\ln(-x)\right) = \frac{-1}{-x} = \frac{1}{x}$$

Thus $\frac{dy}{dx} = \frac{1}{x}$ for all $x \neq 0$.

(b) $\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$.

(c) $y = \log_3(e^x) = x\log_3(e)$

$\frac{dy}{dx} = \log_3(e)$.

(d) $y = \sin x\,(\ln 5 + \ln x)$

$\frac{dy}{dx} = \cos x \ln 5x + \sin x \cdot \frac{1}{x}$

(e) $y = \log_5 x + x\log_5(e)$

$\frac{dy}{dx} = \frac{1}{x\ln 5} + \log_5(e)$.

(f) $y = 5\ln(2x+1) - \frac{1}{2}\ln(x^2+1)$

$\frac{dy}{dx} = 5\cdot\frac{2}{2x+1} - \frac{1}{2}\cdot\frac{2x}{x^2+1} = \frac{10}{2x+1} - \frac{x}{x^2+1}$.

<u>Ex.</u> Find the derivative

(a) $y = x^a$ (b) $y = a^x$ (c) $y = x^x$

(d) $y = x^{\sin x}$ (e) $y = e^{x^x}$ (f) $y = (\ln x)^{\cos x}$

<u>Solution:</u>

(a) $\frac{dy}{dx} = ax^{a-1}$ (b) $\frac{dy}{dx} = a^x \cdot \ln a$.

(c) $\ln y = x\ln x \Rightarrow \frac{1}{y}y' = \ln x + x\cdot\frac{1}{x} = \ln x + 1$

$y' = y(\ln x+1) = x^x(\ln x+1)$.

(d) $\ln y = \sin x \ln x \Rightarrow \frac{1}{y} y' = \cos x \ln x + \frac{\sin x}{x}$

$y' = y\left(\cos x \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

$= x^{\sin x}\left(\ln\left(x^{\cos x}\right) + \frac{\sin x}{x}\right).$

(e) $\ln(y) = x^x \ln e = x^x \Rightarrow \frac{1}{y}\cdot y' = x^x(\ln x + 1)$

$y' = y \cdot x^x(\ln x + 1) = e^{x^x} x^x (\ln x + 1).$

(f) $\ln y = \cos x \ln(\ln x) \Rightarrow \frac{1}{y} y' = -\sin x \ln(\ln x)$

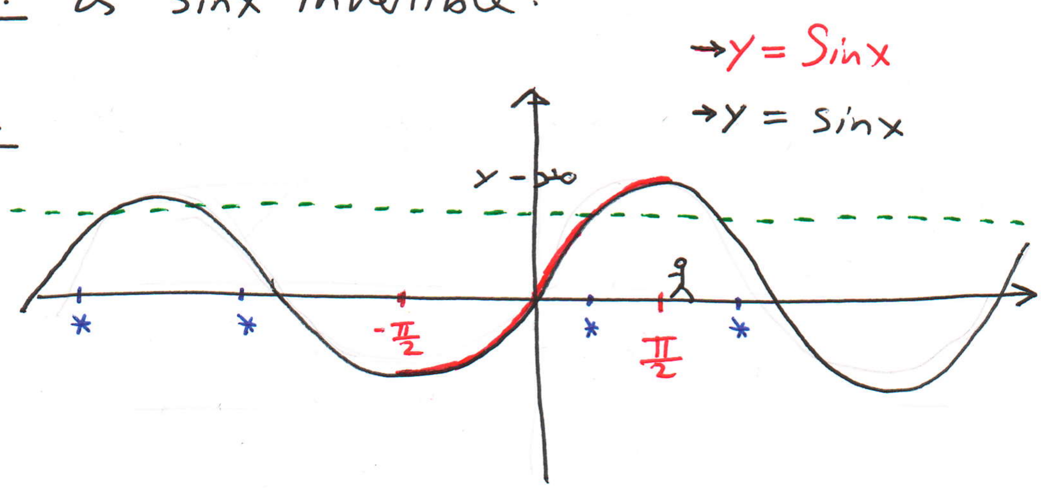
$+ \cos x \frac{1}{\ln x} \cdot \frac{1}{x}$

$y' = (\ln x)^{\cos x}\left(\frac{\cos x}{x \ln x} - \sin x \ln(\ln x)\right)$

Derivatives of Inverse Trig Functions

Q. Is $\sin x$ invertible?

A.



No! Multiple inputs are mapped to the same output.
The graph of $\sin x$ does not pass horizontal line test.

We can, however, restrict $\sin x$ to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and define $\operatorname{Sin} x = \sin x \;;\; x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Then $\operatorname{Sin} x$ is 1-1 and therefore invertible, with inverse $\operatorname{Sin}^{-1} x$. (we will in general keep the distinction between $\sin x$ and $\operatorname{Sin} x$ implicit and write $\sin x$ instead of $\operatorname{Sin} x$ with capital S).

Observe that sin converts angles to ratios and $\sin^{-1}$ converts ratios back to angles.

The astronaut walking on the x-axis sees $y = \sin x$ and the astronaut walking on the y-axis understands the graph as $x = \sin^{-1} y$.

Ex. Compute

(a) $\sin^{-1}(1)$ (b) $\sin^{-1}(-1)$ (c) $\sin^{-1}(\frac{\sqrt{2}}{2})$

(d) $\sin^{-1}(-\frac{1}{2})$ (e) $\sin^{-1}(-\frac{\sqrt{3}}{2})$ (f) $\sin^{-1}(5)$

Solution:

(a) $\sin^{-1}(1)$ is the angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ which sin maps to the ratio 1. Since $\sin(\frac{\pi}{2}) = 1$, it follows that $\sin^{-1}(1) = \frac{\pi}{2}$.

(b) Similarly $\sin^{-1}(-1) = -\frac{\pi}{2}$

(c) $\sin^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$

(d) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

(e) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

(f) $\sin\theta \in [-1,1]$. No angle is carried to 5.

Hence $\sin^{-1}(5) = \emptyset$.

Similarly we can define

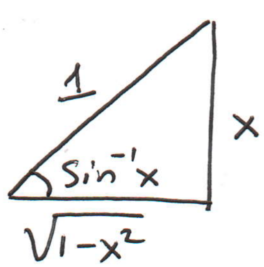
$$\begin{aligned}
&\mathrm{Cos}\, x && x \in [0,\pi] \\
&\mathrm{Sec}\, x && x \in [0,\tfrac{\pi}{2}) \cup (\tfrac{\pi}{2},\pi] \\
&\mathrm{Tan}\, x && x \in (-\tfrac{\pi}{2},\tfrac{\pi}{2}) \\
&\mathrm{Csc}\, x && x \in [-\tfrac{\pi}{2}, 0) \cup (0,\tfrac{\pi}{2}]
\end{aligned}$$

We will not use capital letters and simply note that, say, $\tan^{-1}$ is the inverse of Tan which we continue to write as tan.

<u>Ex.</u> Find $\frac{d}{dx}\mathrm{Sin}^{-1}x$

<u>Solution:</u> $y = \mathrm{Sin}^{-1}x \Rightarrow \mathrm{Sin}\, y = x$

$$\mathrm{Cos}\, y \cdot y' = 1 \Rightarrow y' = \frac{1}{\mathrm{Cos}\, y} = \frac{1}{\mathrm{Cos}(\mathrm{Sin}^{-1}x)} = \frac{1}{\sqrt{1-x^2}}$$

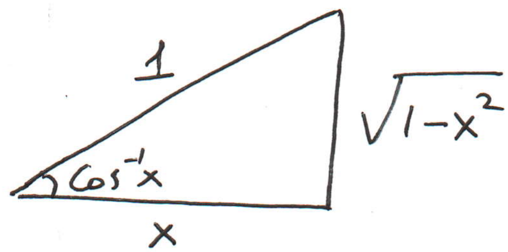


$\mathrm{Cos}(\mathrm{Sin}^{-1}x) = \sqrt{1-x^2}$

Ex. Find $\frac{d}{dx}\cos^{-1}x$.

Solution: $y = \cos^{-1}x \Rightarrow \cos y = x$

$-\sin y \cdot y' = 1$. so $y' = \dfrac{1}{-\sin y} = \dfrac{-1}{\sin(\cos^{-1}x)}$



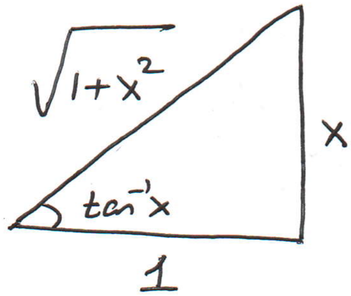
so $\frac{d}{dx}\cos^{-1}x = \dfrac{-1}{\sqrt{1-x^2}}$

Ex. Find $\frac{d}{dx}\tan^{-1}x$

Solution: $y = \tan^{-1}x \Rightarrow \tan y = x$

$\sec^2 y \cdot y' = 1$ so $y' = \dfrac{1}{\sec^2 y} = \dfrac{1}{\sec^2(\tan^{-1}x)}$

$= \dfrac{1}{1+x^2}$.



so $\sec^2(\tan^{-1}x)$

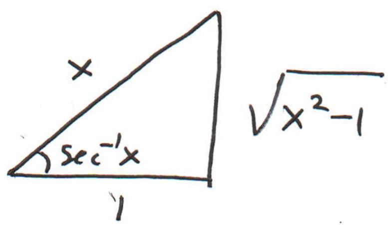
$= \left(\sqrt{1+x^2}\right)^2 = 1+x^2$

Ex. $\frac{d}{dx}\sec^{-1}x = ?$

Solution: $y = \sec^{-1}x \Rightarrow \sec y = x$

$\sec y \tan y \cdot y' = 1$

so $y' = \dfrac{1}{\sec y \tan y} = \dfrac{1}{\sec(\sec^{-1}x)\tan(\sec^{-1}x)} = \dfrac{1}{x\sqrt{x^2-1}}$



Comprehension Check

Verify that $\frac{d}{dx}\csc^{-1}x = \dfrac{-1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}\cot^{-1}x = -\dfrac{1}{1+x^2}$.

Ex. Find the derivative.

(a) $\frac{d}{dx}\left(\tan^{-1}x\right)^2$ (b) $\frac{d}{dx}\tan^{-1}\left(x-\sqrt{1+x^2}\right)$

(c) $h(t) = \cot^{-1}(t) + \cot^{-1}\left(\frac{1}{t}\right)$ (d) $f(v) = \dfrac{v^2 e^{\sin^{-1}v}}{(v^3+5)^6 e^{\sec^{-1}v}}$

(e) $k(x) = \left(\sec^{-1}x\right)^{\tan^{-1}x}$. (f) $\sec^{-1}x + \tan^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$

Solution:

(a) $y = \left(\tan^{-1}x\right)^2$

$$y' = 2\tan^{-1}x \cdot \frac{1}{1+x^2} = \frac{2\tan^{-1}x}{1+x^2}.$$

(b) $y = \tan^{-1}\left(x - \sqrt{1+x^2}\right)$

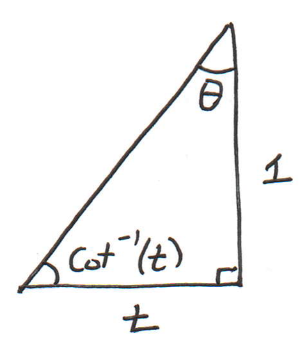
$$y' = \frac{1}{1+\left(x-\sqrt{1+x^2}\right)^2} \cdot \left(1 - \frac{x}{\sqrt{1+x^2}}\right)$$

(c) $h'(t) = -\dfrac{1}{1+t^2} + \dfrac{1}{t^2}\,\dfrac{1}{1+\left(\frac{1}{t}\right)^2}$

$$= -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0$$

so.... $h(t) = \text{constant}$!! ?

Observe that the triangle corresponding to $\cot^{-1}(t)$ is



so $\cot^{-1}(t) + \theta = \frac{\pi}{2}$ Now $\theta = \cot^{-1}\left(\frac{1}{t}\right) = \tan^{-1}(t)$.

(why?). Thus $h(t) = \cot^{-1}(t) + \cot^{-1}\left(\frac{1}{t}\right) = \frac{\pi}{2}$.

(d) set $y = \dfrac{v^2 e^{\sin^{-1} v}}{(v^3+5)^6 e^{\sec^{-1} v}}$ This is unpleasant to differentiate directly, so let's use logarithmic diff.!

$$\ln y = 2\ln v + \sin^{-1} v - 6\ln(v^3+5) - \sec^{-1} v.$$

$$\frac{1}{y} y' = \frac{2}{v} + \frac{1}{\sqrt{1-v^2}} - 6\frac{3v^2}{v^3+5} - \frac{1}{v\sqrt{v^2-1}}$$

so $y' = y\left(\dfrac{2}{v} + \dfrac{1}{\sqrt{1-v^2}} - \dfrac{18v^2}{v^3+5} - \dfrac{1}{v\sqrt{v^2-1}}\right)$

Hence $f'(v) = \dfrac{v^2 e^{\sin^{-1} v}}{(v^3+5)^6 e^{\sec^{-1} v}}\left(\dfrac{2}{v} + \dfrac{1}{\sqrt{1-v^2}} - \dfrac{18v^2}{v^3+5} - \dfrac{1}{v\sqrt{v^2-1}}\right)$

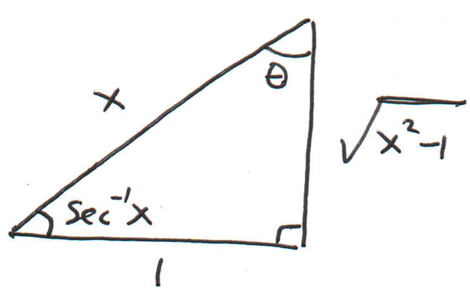
(e) set $y = (\sec^{-1} x)^{\tan^{-1} x}$

$\ln y = \tan^{-1} x \ln(\sec^{-1} x)$; $\dfrac{1}{y}\cdot y' = \dfrac{1}{1+x^2}\ln(\sec^{-1} x)$

$+ \tan^{-1} x \dfrac{1}{\sec^{-1} x}\cdot\dfrac{1}{x\sqrt{x^2-1}}$

or $y' = (\sec^{-1} x)^{\tan^{-1} x}\left(\dfrac{\ln(\sec^{-1} x)}{1+x^2} + \dfrac{\tan^{-1} x}{\sec^{-1}(x)\cdot x\sqrt{x^2-1}}\right)$

(f)



$\theta + \sec^{-1} x = \frac{\pi}{2}$. but $\theta = \tan^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$ $\left(= \sec^{-1}\left(\dfrac{x}{\sqrt{x^2-1}}\right)\right)$

Thus $\dfrac{d}{dx}\left(\sec^{-1} x + \tan^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right)\right) = \dfrac{d}{dx}\left(\dfrac{\pi}{2}\right) = 0.$